# Reverse triple I method of restriction for fuzzy reasoning*

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**Abstract** A theory of reverse triple I method of restriction for implication operator $R_0$ is proposed. And the general computation formulas of infimum for fuzzy modus ponens and supremum for fuzzy modus tollens of $\alpha$-reverse triple I method of restriction are obtained respectively.

**Keywords: fuzzy reasoning, implication operator $R_0$, reverse triple I method of restriction.**

Since Zadeh[1] proposed the method of compositional rule of inference (CRI) for fuzzy reasoning, the method has been generalized respectively from diversified aspects in Refs. [2 ~ 4]. Generally, because this method does not have the reversibility properties, the MP-approximation property of the method of CRI was studied using certain common compositional operators and implication operators in Ref. [5]. Moreover, in order to avoid some drawbacks of the method of CRI, Wang[6] proposed first triple I method with total inference rule that utilizes the implication operator in every step of the reasoning. Afterwards, the theory of restriction degree of triple I method was further presented by the authors[7], its generalization form should be expressed as the following optimal problem.

For any $\alpha \in (0,1]$, $A \in \mathscr{F}(X)$, $B \in \mathscr{F}(Y)$, and $A^* \in \mathscr{F}(X)$ (or $B^* \in \mathscr{F}(Y)$), seek the optimal $B^* \in \mathscr{F}(Y)$ (or $A^* \in \mathscr{F}(X)$) satisfying

$$(A(x) \to B(y)) \to (A^*(x) \to B^*(y)) \leqslant \alpha, \tag{1}$$

for any $x \in X$ and $y \in Y$, where $\mathscr{F}(X)$ and $\mathscr{F}(Y)$ denote respectively the collections consisting of all fuzzy subsets of $X$ and $Y$. And for the implication operator $R_0: [0,1]^2 \to [0,1]$:

$$R_0(a,b) = \begin{cases} 1, & a \leqslant b, \\ a' \vee b, & a > b, \end{cases} \quad (\text{here } a' = 1 - a).$$

General computation formulas of supremum for fuzzy modus ponens (FMP) and infimum for fuzzy modus tollens (FMT) of $\alpha$-triple I method are obtained respectively in Ref. [7].

In this paper, the theory of reverse triple I method of restriction is proposed, its generalization form should be represented as the follows.

Under the hypotheses of (1), seek the optimal $B^* \in \mathscr{F}(Y)$ (or $A^* \in \mathscr{F}(X)$) such that

$$(A^*(x) \to B^*(y)) \to (A(x) \to B(y)) \leqslant \alpha, \tag{2}$$

for any $x \in X$ and $y \in Y$.

The computation formulas of infimum for FMP and supremum for FMT of $\alpha$-reverse triple I method of restriction are given respectively, using the implication operator $R_0$ in this paper.

## 1 Infimum for FMP of $\alpha$-reverse triple I method of restriction

Now, we consider the generalization problems of reverse triple I method of restriction, i.e. for given $\alpha \in (0, 1]$, seek the optimal solution satisfying (2). At first, for the generalization problem of FMP, we give the following principle of restriction of $\alpha$-reverse triple I method.

**Principle of restriction for $\alpha$-reverse triple I FMP**. Suppose that $X$ and $Y$ are non-empty sets, $A, A^* \in \mathscr{F}(X)$, $B \in \mathscr{F}(Y)$. If $B^*$ is the minimal fuzzy set in $\mathscr{F}(Y)$ to satisfy (2), then $B^*$ is called the $\alpha$-solution of (2) for the reverse triple I FMP.

**Remark 1**. For any $y \in Y$ when $B^*(y) \equiv 1$, the left side of (2) will always take its minimal value $R_0(A(x), B(y))$; when $B^*(y) \equiv 0$, it will always

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take its maximal value

$$
\begin{aligned}
&(A^*(x))' \to R_0(A(x),B(y)) \\
&= \begin{cases} 1, & (A^*(x))' \leqslant R_0(A(x),B(y)), \\ A^*(x), & (A^*(x))' > R_0(A(x),B(y)), A^*(x) > R_0(A(x),B(y)), \\ R_0(A(x),B(y)), & (A^*(x))' > R_0(A(x),B(y)), A^*(x) \leqslant R_0(A(x),B(y)). \end{cases}
\end{aligned} \tag{3}
$$

By (3), when

$$(A^*(x))' > R_0(A(x),B(y))$$

and

$$A^*(x) \leqslant R_0(A(x),B(y)), \tag{4}$$

the maximal value of the left side of (2) is $R_0(A(x),B(y))$, and its minimal value is always $R_0(A(x), B(y))$. So, if (4) holds, for any $\alpha \in [R_0(A(x), B(y)),1]$, the minimal fuzzy set in $\mathscr{F}(Y)$ satisfying (2) is $B^*(y)\equiv 0$. If (4) does not hold, for the generalization problem of FMP, the range of $\alpha$ should be confined as

$$\alpha \in (R_0(A(x),B(y)),1). \tag{5}$$

Furthermore, we have the formula of infimum for FMP of $\alpha$-reverse triple I method of restriction as follows.

**Theorem 1. (formula of infimum for FMP of $\alpha$-reverse triple I method of restriction 1)** Suppose that $X$, $Y$ are non-empty sets, $A$ and $A^* \in \mathscr{F}(X)$ and $B \in \mathscr{F}(Y)$. For any $y \in Y$, if $x \in E_y$, then the infimum $B^*(y)$ consisting of fuzzy sets in $\mathscr{F}(Y)$ to satisfy (2) is determined by

$$
\begin{aligned}
B^*(y) = &\sup_{x\in E_y\cap K_y}[A^*(x) \wedge (R_0(A(x),B(y)) \vee \alpha')] \\
&\cdot \chi_{E_y\cap K_y} \\
&+ \sup_{x\in E_y-K_y}[A^*(x) \wedge (R_0(A(x),B(y))] \\
&\cdot \chi_{E_y-K_y},
\end{aligned} \tag{6}
$$

where $K_y = \{x\in X \mid A^*(x) > \alpha\}$ and $E_y = \{x \in X \mid (A^*(x))' \leqslant R_0(A(x),B(y))\}$.

**Proof**. For any $y\in Y$, we will first prove: any $C(y) \in \mathscr{F}(Y)$ with $C(y) > B^*(y)$ must satisfy (2). For $x \in E_y$, we will discuss in different cases as follows.

**Case 1**. If $x \in E_y \cap K_y$, then using $B^*(y)$ determined by (6), we have

$$
\begin{aligned}
C(y) &> B^*(y) \\
&\geqslant A^*(x) \wedge (R_0(A(x),B(y)) \vee \alpha').
\end{aligned} \tag{7}
$$

Further, the discussion will be partitioned again into two possible cases.

(i) If $A^*(x) \leqslant C(y)$ then $R_0(A^*(x), C(y))\equiv 1$, consequently

$$
\begin{aligned}
M_{xy} &= R_0(A^*(x),C(y)) \to R_0(A(x),B(y)) \\
&= 1 \to R_0(A(x),B(y)) \\
&= R_0(A(x),B(y)) \leqslant \alpha.
\end{aligned}
$$

(ii) If $A^*(x) > C(y)$, then $R_0(A^*(x), C(y)) = (A^*(x))' \vee C(y)$. By (7), we have

$$C(y) > B^*(y) \geqslant R_0(A(x),B(y)) \vee \alpha'. \tag{8}$$

This implies that $(C(y))' < \alpha$. Moreover, by the hypotheses in (8) and (5), we have

$$
\begin{aligned}
M_{xy} &= R_0(A^*(x),C(y)) \to R_0(A(x),B(y)) \\
&= (A^*(x) \wedge (C(y))') \vee R_0(A(x),B(y)) \\
&< \alpha.
\end{aligned} \tag{9}
$$

**Case 2**. If $x \in E_y - K_y$, similarly, from the sense of $B^*(y)$ determined by (6), we know

$$C(y) > B^*(y) \geqslant A^*(x) \wedge R_0(A(x),B(y)). \tag{10}$$

The discussion will be partitioned into two possible cases.

(i) If $A^*(x) \leqslant C(y)$, then from the proof of (i) in Case 1 of Theorem 1, we know that $C(y)$ satisfies (2).

(ii) If $A^*(x) > C(y)$, then $R_0(A^*(x), C(y)) = (A^*(x))' \vee C(y)$. From (10), we have

$$C(y) > B^*(y) \geqslant R_0(A(x),B(y)). \tag{11}$$

From this, and noting that $x \notin K_y$, we deduce $A^*(x) \leqslant \alpha$. In addition, using the hypothesis in (5), we get

$$
\begin{aligned}
M_{xy} &= R_0(A^*(x),C(y)) \to R_0(A(x),B(y)) \\
&= (A^*(x) \wedge (C(y))') \vee R_0(A(x),B(y)) \\
&\leqslant \alpha.
\end{aligned} \tag{12}
$$

Combining the proofs of Case 1 and Case 2, it follows that $C(y)$ satisfies (2). On the other hand, we will prove: for some $y_0 \in Y$ and any $D(y_0) \in \mathscr{F}(Y)$ with $D(y_0) < B^*(y_0)$, $D(y_0)$ cannot satisfy (2). In fact, applying the sense of $B^*(y_0)$ determined by (6), the discussion will be partitioned into two possible cases.

**Case 1**. There exists $x_0 \in E_{y_0} \cap K_{y_0}$ such that

$$D(y_0) < A^*(x_0) \wedge (R_0(A(x_0), B(y_0)) \vee \alpha'). \quad (13)$$

We will discuss again in two possible cases.

(i) If $D(y_0) \leqslant R_0(A(x_0), B(y_0))$. Then

$$R_0(A^*(x_0), D(y_0)) \to R_0(A(x_0), B(y_0)) = ((A^*(x_0))' \vee D(y_0)) \to R_0(A(x_0), B(y_0)) = 1 > \alpha.$$

(ii) If $D(y_0) > R_0(A(x_0), B(y_0))$, then from (13), it is known that $D(y_0) < \alpha'$. Noting that $x_0 \in K_{y_0}$, we have

$$R_0(A^*(x_0), D(y_0)) \to R_0(A(x_0), B(y_0)) = ((A^*(x_0))' \vee D(y_0)) \to R_0(A(x_0), B(y_0)) = (A^*(x_0) \wedge (D(y_0))' \vee R_0(A(x_0), B(y_0)) > \alpha.$$

**Case 2**. There exists $x_0 \in E_{y_0} - K_{y_0}$ such that

$$D(y_0) < A^*(x_0) \wedge R_0(A(x_0), B(y_0)). \quad (14)$$

Consequently, we have

$$R_0(A^*(x_0), D(y_0)) \to R_0(A(x_0), B(y_0)) = ((A^*(x_0))' \vee D(y_0)) \to R_0(A(x_0), B(y_0)) = 1 > \alpha. \quad (15)$$

So, $D(y_0)$ cannot satisfy (2). All of these show that $B^*(y)$ is the infimum consisting of fuzzy sets in $\mathscr{F}(Y)$ to satisfy (2).

**Theorem 2. (formula of infimum for FMP of $\alpha$-reverse triple I method of restriction 2)** Suppose that $X$, $Y$ are non-empty sets, $A$ and $A^* \in \mathscr{F}(X)$ and $B \in \mathscr{F}(Y)$. For any $y \in Y$, if $x \in F_y$, then the infimum $B^*(y)$ consisting of fuzzy sets in $\mathscr{F}(Y)$ to satisfy (2) is determined by

$$B^*(y) = \sup_{x \in F_y \cap K_y} [A^*(x) \wedge \alpha']. \quad (16)$$

Here $K_y = \{x \in X \mid A^*(x) > \alpha\}$ and $F_y = \{x \in X \mid A^*(x) \wedge (A^*(x))' > R_0(A(x), B(y))\}$.

**Proof**. For any $y \in Y$ and $B^*(y)$ determined by (16), we will first prove: any $C(y) \in \mathscr{F}(Y)$ with $C(y) > B^*(y)$ must satisfy (2). For $x \in F_y$, we will discuss in different cases as follows.

**Case 1**. If $x \in F_y \cap K_y$, then applying $B^*(y)$ determined by (16), we know

$$C(y) > B^*(y) \geqslant A^*(x) \wedge \alpha'. \quad (17)$$

We will discuss again in two possible cases.

(i) If $A^*(x) \leqslant C(y)$, then from the proof of (i) in Case 1 of Theorem 1, we know that $C(y)$ satisfies (2).

(ii) If $A^*(x) > C(y)$, then $R_0(A^*(x), C(y)) = (A^*(x))' \vee C(y)$. Noting (17), we have $C(y) > B^*(y) \geqslant \alpha'$. It follows that $(C(y))' < \alpha$. By the hypothesis in Case 1 of Theorem 2, and the hypothesis in (5), we have

$$M_{xy} = R_0(A^*(x), C(y)) \to R_0(A(x), B(y)) = (A^*(x) \wedge (C(y))') \vee R_0(A(x), B(y)) < \alpha. \quad (18)$$

**Case 2**. If $x \in F_y - K_y$, similarly, from the sense of $B^*(y)$ determined by (16), we know $B^*(y) = 0$. Noting that $x \in F_y$, $x \notin K_y$, and the hypothesis in (5), we have

$$M_{xy} = R_0(A^*(x), B^*(y)) \to R_0(A(x), B(y)) = (A^*(x))' \to R_0(A(x), B(y)) = A^*(x) \vee R_0(A(x), B(y)) \leqslant \alpha. \quad (19)$$

Combining the proofs of Case 1 and Case 2, we know that $C(y)$ satisfies (2). On the other hand, for some $y_0 \in Y$ with $B^*(y_0) > 0$, any $D(y_0)$ provided $D(y_0) < B^*(y_0)$ cannot satisfy (2). In fact, by the sense of $B^*(y_0)$ determined by (16), there exists $x_0 \in F_{y_0} \cap K_{y_0}$ such that $D(y_0) < A^*(x_0) \wedge \alpha'$. So, noting that $x_0 \in F_{y_0}$ and $x_0 \in K_{y_0}$, and the hypothesis in (5), we obtain

$$R_0(A^*(x_0), D(y_0)) \to R_0(A(x_0), B(y_0)) = ((A^*(x_0))' \vee D(y_0)) \to R_0(A(x_0), B(y_0)) = (A^*(x_0) \wedge (D(y_0))' \vee R_0(A(x_0), B(y_0)) > \alpha.$$

So, $D(y_0)$ cannot satisfy (2). Sum up the above proof, $B^*(y) \in \mathscr{F}(Y)$ is the infimum consisting of fuzzy sets to satisfy (2).

## 2 Supremum for FMT of $\alpha$-reverse triple I method of restriction

Now, let us consider the generalization problem of FMT, we give the following principle of restriction of the $\alpha$-reverse triple I method.

**Principle of restriction for $\alpha$-reverse triple I FMT**. Suppose that $X$ and $Y$ are non-empty sets, $A \in \mathscr{F}(X)$, $B$ and $B^* \in \mathscr{F}(Y)$. If $A^*$ is the maximal fuzzy set in $\mathscr{F}(X)$ to satisfy (2), then $A^*$ is called the $\alpha$-solution of (2) for the reverse triple I FMT.

**Remark 2**. For any $x \in X$, when $A^*(x) \equiv 0$, the left side of (2) will always take its minimal value

$R_0(A(x),B(y))$; and when $A^*(x)\equiv 1$, it will always take its maximal value

$$B^*(y)\to R_0(A(x),B(y)) = \begin{cases} 1, & B^*(y)\leqslant R_0(A(x),B(y)), \\ R_0(A(x),B(y)), & B^*(y)>R_0(A(x),B(y)),(B^*(y))'\leqslant R_0(A(x),B(y)), \\ (B^*(y))', & B^*(y)>R_0(A(x),B(y)),(B^*(y))'>R_0(A(x),B(y)). \end{cases} \quad (20)$$

By (20), when

$$(B^*(y))' > R_0(A(x),B(y))$$

and

$$(A^*(y))' \leqslant R_0(A(x),B(y)), \quad (21)$$

the maximal value of the left side of (2) is $R_0(A(x),B(y))$, and its minimal value is always $R_0(A(x),B(y))$. Hence, if (21) holds, for any $\alpha\in[R_0(A(x),B(y)),1]$, the maximal fuzzy set in $\mathscr{F}(X)$ satisfying (2) is $A^*(x)\equiv 1$.

If (21) does not hold, for the generalization problem of FMT, the range of $\alpha$ should also be confined by (5).

Furthermore, we have the formula of supremum for FMT of $\alpha$-reverse triple I method of restriction as follows.

**Theorem 3. (formula of supermum for FMT of $\alpha$-reverse triple I method of restriction 1)** Suppose that $X$, $Y$ are non-empty sets, $A\in\mathscr{F}(X)$, $B$ and $B^*\in\mathscr{F}(Y)$. For any $x\in X$, if $y\in E_x$, then the supremum $A^*(x)$ consisting of fuzzy sets in $\mathscr{F}(X)$ to satisfy (2) is determined by

$$A^*(x)=\inf_{y\in E_x\cap K_x}[B^*(y)\vee(R_0'(A(x),B(y))\wedge\alpha)]\cdot\chi_{E_x\cap K_x} + \inf_{y\in E_x-K_x}[B^*(y)\vee R_0'(A(x),B(y))]\cdot\chi_{E_x-K_x}, \quad (22)$$

where $E_x=\{y\in Y\mid B^*(y)\leqslant R_0(A(x),B(y))\}$ and $K_x=\{y\in Y\mid B^*(y)<\alpha'\}$.

**Proof**. For any $x\in X$ and $C(x)<A^*(x)$, we will first prove: $C(x)$ must satisfy (2). For $y\in E_x$, we will discuss in different cases as follows.

**Case 1**. If $y\in E_x\cap K_x$, then from $A^*(x)$ determined by (22), we have

$$C(x)<A^*(x) \leqslant B^*(y)\vee(R_0'(A(x),B(y))\wedge\alpha). \quad (23)$$

Further, the discussion will be partitioned again into two possible cases.

(i) If $C(x)\leqslant B^*(y)$, then $R_0(C(x),B^*(y))=1$, similar to the proof of (1) in Case 1 of Theorem 1, we get

$$M_{xy}=R_0(C(x),B^*(y))\to R_0(A(x),B(y))\leqslant\alpha.$$

(ii) If $C(x)>B^*(y)$, then $R_0(C(x),B^*(y))=C'(x)\vee B^*(y)$. Noting (23), we have

$$C(x)<A^*(x)\leqslant R_0'(A(x),B(y))\wedge\alpha. \quad (24)$$

This implies $C(x)<\alpha$ and $C'(x)>R_0(A(x),B(y))$. Consequently, using the hypothesis in (5), we deduce

$$M_{xy}=R_0(C(x),B^*(y))\to R_0(A(x),B(y)) =(C(x)\wedge(B^*(y))')\vee R_0(A(x),B(y)) <\alpha. \quad (25)$$

**Case 2**. If $y\in E_x-K_x$, similarly, from the sense of $A^*(x)$ determined by (22), we have

$$C(x)<A^*(x)\leqslant B^*(y)\vee R_0'(A(x),B(y)). \quad (26)$$

We will discuss again in two possible cases.

(i) If $C(x)\leqslant B^*(y)$, then from (1) in Case 1 of Theorem 3, $C(x)$ satisfies (2).

(ii) If $C(x)>B^*(y)$, then $R_0(C(x),B^*(y))=C'(x)\vee B^*(y)$. By (26), we have $C(x)<R_0'(A(x),B(y))$. It follows that $C'(x)>R_0(A(x),B(y))$. Noting $y\notin K_x$ and the hypothesis in (5), we have

$$M_{xy}=R_0(C(x),B^*(y))\to R_0(A(x),B(y)) =(C(x)\wedge(B^*(y))')\vee R_0(A(x),B(y)) \leqslant\alpha. \quad (27)$$

Combining the discussions of Case 1 and Case 2, we know that $C(x)$ satisfies (2). On the other hand, for some $x_0\in X$ and any $D(x_0)\in\mathscr{F}(X)$ with $D(x_0)>A^*(x_0)$, $D(x_0)$ cannot satisfy (2). In fact, using the sense of $A^*(x_0)$ determined by (22), the discussion will be partitioned again into two possible cases.

**Case 1**. There exists $y_0\in E_{x_0}\cap K_{x_0}$ such that

$$D(x_0)>B^*(y_0)\vee(R_0'(A(x_0),B(y_0))\wedge\alpha). \quad (28)$$

Then, we will discuss in two possible cases.

(i) If $D(x_0) \geqslant R_0'(A(x_0), B(y_0))$, then

$$R_0(D(x_0), B^*(y_0)) \to R_0(A(x_0), B(y_0)) = (D(x_0))' \vee B^*(y_0) \to R_0(A(x_0), B(y_0)) = 1 > \alpha.$$

(ii) If $D(x_0) < R_0'(A(x_0), B(y_0))$, then (28) yields $D(x_0) > \alpha$. By $y_0 \in K_{x_0}$, we have

$$R_0(D(x_0), B^*(y_0)) \to R_0(A(x_0), B(y_0)) = (D(x_0) \wedge (B^*(y_0))') \vee R_0(A(x_0), B(y_0)) > \alpha.$$

**Case 2**. There exists $y_0 \in E_{x_0} - K_{x_0}$ such that $D(x_0) > B^*(y_0) \vee R_0'(A(x_0), B(y_0))$. Then $D(x_0) > R_0'(A(x_0), B(y_0))$. By noting $y_0 \in E_{x_0}$, we get

$$R_0(D(x_0), B^*(y_0)) \to R_0(A(x_0), B(y_0)) = (D(x_0))' \vee B^*(y_0) \to R_0(A(x_0), B(y_0)) = 1 > \alpha.$$

So, $D(x_0)$ cannot satisfy (2). To sum up, $A^*(x)$ is the supremum consisting of fuzzy sets in $\mathscr{F}(X)$ to satisfy (2).

**Theorem 4. (formula of supremum for FMT of $\alpha$-reverse triple I method of restriction 2)** Suppose that $X$, $Y$ are non-empty sets, $A \in \mathscr{F}(X)$, $B$ and $B^* \in \mathscr{F}(Y)$. For any $x \in X$, if $y \in F_x$, then the supremum $A^*(x)$ consisting of fuzzy sets in $\mathscr{F}(X)$ to satisfy (2) is determined by

$$A^*(x) = \inf_{y \in F_x \cap K_x} [B^*(y) \vee \alpha], \qquad (29)$$

where $F_x = \{y \in Y \mid B^*(y) \wedge (B^*(y))' > R_0(A(x), B(y))\}$ and $K_x = \{y \in Y \mid B^*(y) < \alpha'\}$.

**Proof**. For any $x \in X$ and $C(x) < A^*(x)$, we will first prove: $C(x)$ must satisfy (2). For $y \in F_x$ we will discuss in different cases as follows.

**Case 1**. If $y \in F_x \cap K_x$, then from $A^*(x)$ determined by (29), we have

$$C(x) < A^*(x) \leqslant B^*(y) \vee \alpha. \qquad (30)$$

We will discuss again in two possible cases.

(i) If $C(x) \leqslant B^*(y)$, then from the proof of (1) in Case 1 of Theorem 3, we know that $C(y)$ satisfies (2).

(ii) If $C(x) > B^*(y)$, then $R_0(C(x), B^*(y)) = C'(x) \vee B^*(y)$. Noting (30), we have $C(x) < A^*(x) \leqslant \alpha$. From this, and applying the hypothesis of Case 1 of Theorem 4, and the hypothesis in (5), we get

$$M_{xy} = R_0(C(x), B^*(y)) \to R_0(A(x), B(y)) = (C(x) \wedge (B^*(y))') \vee R_0(A(x), B(y)) < \alpha. \qquad (31)$$

**Case 2**. If $y \in F_x - K_x$, then using $A^*(x)$ determined by (29), we have $A^*(x) = 1$. Noting $y \in F_x$ and $y \notin K_x$ the hypothesis in (5), we obtain

$$M_{xy} = R_0(A^*(x), B^*(y)) \to R_0(A(x), B(y)) = B^*(y) \to R_0(A(x), B(y)) = (B^*(y))' \vee R_0(A(x), B(y)) = (B^*(y))' \leqslant \alpha. \qquad (32)$$

Combining the discussions of Case 1 and Case 2, we know that $C(x)$ satisfies (2). On the other hand, for some $x_0 \in X$ and any $D(x_0) \in \mathscr{F}(X)$ with $D(x_0) > A^*(x_0)$, $D(x_0)$ cannot satisfy (2). In fact, using the sense of $A^*(x_0)$ determined by (29), there exists $y_0 \in F_{x_0} \cap K_{x_0}$ such that $D(x_0) > B^*(y_0) \vee \alpha$. Moreover, by $y_0 \in F_{x_0}$ and $y_0 \in K_{x_0}$, we have

$$R_0(D(x_0), B^*(y_0)) \to R_0(A(x_0), B(y_0)) = (D(x_0))' \vee B^*(y_0)) \to R_0(A(x_0), B(y_0)) = (D(x_0) \wedge (B^*(y_0))') \vee R_0(A(x_0), B(y_0)) > \alpha.$$

So, $D(x_0)$ cannot satisfy (2). Sum up the above proof, $A^*(x)$ is the supremum consisting of fuzzy sets in $\mathscr{F}(X)$ to satisfy (2).

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